1. Introduction

Griffith theory [18] is a model explaining the quasi-static crack growth in elastic bodies under the assumption that the crack set is preassigned. In a two-dimensional setting, let us denote by $\Omega \subset \mathbb{R}^2$ the reference configuration of a linearly elastic body allowing for cracks inside $\widehat{\Gamma}$. To fix the ideas, provided the evolution is sufficiently smooth, that $\widehat{\Gamma}$ is a simple curve, and that the evolution is growing only in one direction, then the crack is completely characterized by the position of its tip, and thus by its arc length. Denoting by $\Gamma(\ell)$ the crack of length ℓ inside $\widehat{\Gamma}$, the elastic energy associated to a given kinematically admissible displacement $u : \Omega \smallsetminus \Gamma(\ell) \to \mathbb{R}^2$ satisfying $u = \psi(t)$ on $\partial\Omega \smallsetminus \Gamma(\ell)$, is given by

$$E(t; u, \ell) := \frac{1}{2} \int_{\Omega \smallsetminus \Gamma(\ell)} \mathbb{C}e(u) : e(u)\, dx,$$

where $\mathbb{C}$ is the fourth order Hooke's tensor, $e(u)$ is the symmetrized gradient of u, and $\psi(t) : \partial\Omega \to \mathbb{R}^2$ is a prescribed boundary datum depending on time, which is the driving mechanism of the process. If the evolution is slow enough, it is reasonable to neglect inertia and viscous effects so that the quasi-static assumption becomes relevant: at each time t, the body is in elastic equilibrium. It enables one to define the potential energy as

$$\mathscr{P}(t, \ell) := E(t; u(t, \ell), \ell) = \min E(t; \cdot, \ell),$$

where the minimum is computed over all kinematically admissible displacements at time t. Therefore, given a cracking state, the quasi-static assumption permits to find the displacement. In order to get the crack itself (or equivalently its length), Griffith introduced a criterion whose fundamental ingredient is the *energy release rate*. It is defined as the variation of potential energy along an infinitesimal crack increment, or in other words, the quantity of released potential energy with respect to a small crack increment. More precisely, it is given by

$$G(t, \ell) := -\frac{\partial \mathscr{P}}{\partial \ell}(t, \ell)$$

provided the previous expression makes sense. From a thermodynamical point of view, the energy release rate is nothing but the thermodynamic force associated to the crack length (the natural internal variable modeling the dissipative effect of fracture). Griffith's criterion is summarized into the three following items: for each $t > 0$

(i) $G(t, \ell(t)) \leqslant G_c$, where $G_c > 0$ is a characteristic material constant referred to as the toughness of the body;
(ii) $\dot{\ell}(t) \geqslant 0$;
(iii) $\big(G(t, \ell(t)) - G_c\big)\dot{\ell}(t) = 0$.

Item (i) is a threshold criterion which stipulates that the energy release rate cannot exceed the critical value G_c. Item (ii) is an irreversibility criterion which ensures that the crack can only grow. The third and last item is a compatibility condition between (i) and (ii): it states that a crack will grow if and only if the energy release rate constraint is saturated.

In [17] (see also [3]), it has been observed that Griffith's criterion is nothing but the necessary first order optimality condition of a variational model. More precisely, if for every $t>0$, $(u(t),\ell(t))$ satisfies:

(i) *Unilateral minimality:* for any $\widehat{\ell}\geqslant \ell(t)$, and any $v:\Omega\smallsetminus\Gamma(\widehat{\ell})\to\mathbb{R}^2$ satisfying $v=\psi(t)$ on $\partial\Omega\smallsetminus\Gamma(\widehat{\ell})$, then

$$\mathscr{E}(t):=\frac{1}{2}\int_{\Omega\smallsetminus\Gamma(\ell(t))}\mathbb{C}e(u(t)):e(u(t))\,dx+G_c\,\ell(t)\leqslant\frac{1}{2}\int_{\Omega\smallsetminus\Gamma(\widehat{\ell})}\mathbb{C}e(v):e(v)\,dx+G_c\,\widehat{\ell};$$

(ii) *Irreversibility:* $\dot{\ell}(t)\geqslant 0$;

(iii) *Energy balance:*

$$\dot{\mathscr{E}}(t)=\int_{\partial\Omega\smallsetminus\Gamma(\ell(t))}(\mathbb{C}e(u(t))\nu)\cdot\dot{\psi}(t)\,d\mathscr{H}^1,$$

then $(u(t),\ell(t))$ is a solution of Griffith's model. In the previous expression, $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure. The energy balance is nothing but a reformulation of the second law of thermodynamics which asserts the non-negativity of the mechanical dissipation. It states that the temporal variation of the total energy (the sum of the elastic and surface energies) is compensated by the power of external forces, which in our case reduces to the stress $(\mathbb{C}e(u(t))\nu$ acting on $\partial\Omega\smallsetminus\Gamma(\ell(t))$ and generated by the boundary displacement $\psi(t)$. This new formulation relies on the constrained minimization of the total energy of Mumford-Shah type

$$\mathscr{E}(u,\Gamma):=\frac{1}{2}\int_{\Omega\smallsetminus\Gamma}\mathbb{C}e(u):e(u)\,dx+G_c\mathscr{H}^1(\Gamma)$$

which puts in competition a bulk (elastic) energy and a surface (Griffith) energy. One of the main interests is that it makes it possible to get rid of the assumption of the a priori knowledge of the crack path. Following [17], a quasi-static evolution is defined as a mapping $t\mapsto(u(t),\Gamma(t))$ satisfying

(i) *Unilateral minimality:* for any $\overline{\Omega}\supset\widehat{\Gamma}\supset\Gamma(t)$, and any $v:\Omega\smallsetminus\widehat{\Gamma}\to\mathbb{R}^2$ satisfying $v=\psi(t)$ on $\partial\Omega\smallsetminus\widehat{\Gamma}$, then

$$\mathscr{E}(u(t),\Gamma(t))\leqslant\mathscr{E}(v,\widehat{\Gamma});$$

(ii) *Irreversibility:* $\Gamma(s)\subset\Gamma(t)$ for every $s\leqslant t$;

(iii) *Energy balance:*

$$\mathscr{E}(u(t),\Gamma(t))=\mathscr{E}(u(0),\Gamma(0))+\int_0^t\int_{\Omega\smallsetminus\Gamma(s)}\mathbb{C}e(u(s)):e(\dot{\psi}(s))\,dx\,ds.$$

An existence result for this model has been given in [5] (see also [13, 16, 12] in other contexts) for cracks belonging to the class of compact and connected subsets of $\overline{\Omega}$. The main reason of this assumption was to ensure the lower semicontinuity of the Mumford-Shah type functional $(u,\Gamma)\mapsto\mathscr{E}(u,\Gamma)$ with respect to a reasonable notion of convergence. The lower semicontinuity of the surface energy with respect to the Hausdorff convergence of cracks is a consequence of Gołab's Theorem (see [15]), while the continuity of the bulk energy is a consequence of continuity results of the Neumann problem with respect to the Hausdorff convergence of the boundary (see [4, 6]) together with a density result [5]. In any cases, all these results only hold in dimension 2 and in the class of compact and connected sets.

If one is interested into fine qualitative results such as crack initiation (see [8]) of kinking (see [7]) it becomes necessary to understand the nature of the singularity at the crack tip. Therefore one should be able to make rigorous a suitable notion energy release rate. The first proof of the differentiable character of the potential energy with respect to the crack length has been given in [14] (see also [22, 28, 27]). The generalized variational setting described above, a mathematical justification of the notions of energy release rate for any incremental crack attached to a given initial crack has been in [7] in the case where the crack is straight in a small neighborhood of its tip. In the footstep of that work, we attempt here weaken the regularity assumption on the initial crack, which is merely closed, connected, with density 1/2 at the origin (that imply to blow up as a segment at the origin, up to rotations).

1.1. Main Results. — Our main results are contained in Theorem 6.4 and Theorem 7.1 respectively in Section 6 and Section 7.

1.1.1. *First Result.* — The first main result Theorem 6.4 is a purely P.D.E. result. We analyze the blow-up limit of the optimal displacement at the tip of the given initial crack. We prove that for some suitable subsequence, the blow-up limit converges to the classical crack-tip function in the complement of a half-line, i.e., of the form

$$\kappa_1\phi_1 + \kappa_2\phi_2, \tag{1.1}$$

for some constants κ_1 and $\kappa_2 \in \mathbb{R}$, while ϕ_1 and ϕ_2 are positively 1/2-homogenous functions which are explicitly given by (6.15) and (6.16) below.

This part can be seen as a partial generalization in planar elasticity of what was previously done in the anti-plane case [9]. Mathematically speaking, the corresponding function to be studied is now a vectorial function satisfying a Lamé type system, instead of being simply a scalar valued harmonic function. One of the key obstacles in the vectorial case is that no monotonicity property is known for such a problem, which leads to a slightly weaker result than in the scalar case: the convergence of the blow-up sequence only holds up to subsequences, and nothing is known for the whole sequence. Consequently, the constants κ_1 and κ_2 in (1.1) a priori depend on this particular subsequence. As a matter of fact, this prevents us to define properly the stress intensity factor analogously to what was proposed in [9]. On the other hand, we believe that the techniques employed in the proof and the results on their own are already interesting. In addition, the absence of monotonicity is not the only difference with the scalar case, which led us to find a new proof relying on a duality approach via the so-called Airy function in order to bypass some technical problems.

Another substantial difference with the scalar case appears while studying homogeneous solutions of the planar Lamé system in the complement of a half-line, which is crucial in the understanding of blow-up solutions at the crack tip. For harmonic functions it is rather easy to decompose any solutions as a sum of spherical-harmonics directly by writing the operator in polar coordinates, and identify the degree of homogeneity of each term with the corresponding eigenvalue of the Dirichlet-Laplace-Beltrami operator on the circle minus a point. For the Lamé system, or alternatively for the biharmonic equation, a similar naive approach cannot work. The appropriate eigenvalue problem on the circle have a more complicate nature, and analogous results rely on an abstract theory developed first by Kondrat'ev which rests on pencil operators, weighted Sobolev spaces, the Fredholm alternative, and calculus of residues. We used this technology in the proof of Proposition 6.3 for which we could not find a more elementary argument.

1.1.2. *Second result.* — The second main result Theorem 7.1 concerns the energy release rate of an incremental crack Γ, which is roughly speaking the derivative of the elastic energy with respect to the crack increment (see (7.1) for the precise definition). We prove that the value of this limit is realized as an explicit minimization problem in the cracked-plane $\mathbb{R}^2\smallsetminus\big((-\infty,0]\times\{0\}\big)$. One can find a similar statement in [7, Th. 3.1], but with the additional assumption that the initial crack is a line segment close to the origin. We remove here this hypothesis, establishing the same result for any initial crack which is closed, connected and admits a line segment as blow-up limit at the origin. The starting point for this generalization is the knowledge of the blow-up limit at the origin for displacement associated to a general initial crack, namely our first result Theorem 6.4. Since this result holds only up to subsequences, the same restriction appears in the statement of Theorem 6.4 as well.

Therewith, it should be mentioned that Theorem 7.1 is new even for the scalar case, for which the conclusion is even more accurate. Indeed in this case, the monotonicity formula of [9] ensures that the convergence holds for the whole sequence and not only for a subsequence.

The paper is organized as follows: after introducing the main notation in Section 2, we describe precisely the mechanical model in Section 3. Section 4 is devoted to establish technical results related to the existence of the harmonic conjugate and the Airy function associated to the displacement in a neighborhood of the crack tip. In Section 5, we prove lower and upper bounds of the energy release rate. The blow-up analysis of the displacement around the crack tip is the object of Section 6. Section 7 is devoted to give a formula for the energy release rate as a global minimization problem. Finally, we shortly review Kondrat'ev theory of elliptic regularity vs singularity inside corner domains in an appendix.

Acknowledgements. — The authors wish to thank Svitlana Mayboroda for useful discussions about the subject of this paper, and for having pointed out reference [24]. They are also grateful to Monique Dauge for having sent them a copy of the paper [23], and for the argument leading to the proof of Proposition 6.3. The authors wish to thank the referees for their very careful reading of the manuscript, and many comments which helped us to substantially improve the presentation.

2. Mathematical preliminaries

2.1. General notation. — The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathscr{L}^n$, and the k-dimensional Hausdorff measure by $\mathscr{H}^k$. If E is a measurable set, we will sometimes write $|E|$ instead of $\mathscr{L}^n(E)$. If a and $b \in \mathbb{R}^n$, we write $a \cdot b = \sum_{i=1}^n a_i b_i$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open ball of center x and radius ϱ is denoted by $B_\varrho(x)$. If $x = 0$, we simply write B_ϱ instead of $B_\varrho(0)$.

We write $\mathbb{M}^{n\times n}$ for the set of real $n \times n$ matrices, and $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ for that of all real symmetric $n\times n$ matrices. Given a matrix $A \in \mathbb{M}^{n\times n}$, we let $|A| := \sqrt{\operatorname{tr}(AA^T)}$ (A^T is the transpose of A, and $\operatorname{tr} A$ is its trace) which defines the usual Euclidean norm over $\mathbb{M}^{n\times n}$. We recall that for any two vectors a and $b \in \mathbb{R}^n$, $a \otimes b \in \mathbb{M}^{n\times n}$ stands for the tensor product, i.e., $(a\otimes b)_{ij} = a_i b_j$ for all $1 \leqslant i, j \leqslant n$, and $a \odot b := \frac{1}{2}(a\otimes b + b\otimes a) \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ denotes the symmetric tensor product.

Given an open subset U of $\mathbb{R}^n$, we denote by $\mathscr{M}(U)$ the space of all real-valued Radon measures with finite total variation. We use standard notation for Lebesgue spaces $L^p(U)$ and Sobolev spaces $W^{k,p}(U)$ or $H^k(U) := W^{k,2}(U)$. If Γ is a closed subset of $\overline{U}$, we denote by $H^k_{0,\Gamma}(U)$ the closure of $\mathscr{C}^\infty_c(\overline{U} \smallsetminus \Gamma)$ in $H^k(U)$. In particular, if $\Gamma = \partial U$, then $H^k_{0,\partial U}(U) = H^k_0(U)$.

2.2. Capacities. — In the sequel, we will use the notion of capacity for which we refer to [1, 21]. We just recall the definition and several facts. The $(k,2)$-capacity of a compact set $K \subset \mathbb{R}^n$ is defined by

$$\mathrm{Cap}_{k,2}(K) := \inf\left\{ \|\varphi\|_{H^k(\mathbb{R}^n)} : \varphi \in \mathscr{C}^\infty_c(\mathbb{R}^n),\ \varphi \geqslant 1 \text{ on } K \right\}.$$

This definition is then extended to open sets $A \subset \mathbb{R}^n$ by

$$\mathrm{Cap}_{k,2}(A) := \sup\left\{ \mathrm{Cap}_{k,2}(K) : K \subset A,\ K \text{ compact}\right\},$$

and to arbitrary sets $E \subset \mathbb{R}^n$ by

$$\mathrm{Cap}_{k,2}(E) := \inf\left\{ \mathrm{Cap}_{k,2}(A) : E \subset A,\ A \text{ open}\right\}.$$

One of the interests of capacity is that it enables one to give an accurate sense to the pointwise value of Sobolev functions. More precisely, every $u \in H^k(\mathbb{R}^n)$ has a $(k,2)$-quasicontinuous representative $\widetilde{u}$, which means that $\widetilde{u} = u$ a.e. and that, for each $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subset \mathbb{R}^n$ such that $\mathrm{Cap}_{k,2}(\mathbb{R}^n \smallsetminus A_\varepsilon) < \varepsilon$ and $\widetilde{u}|_{A_\varepsilon}$ is continuous on A_ε (see [1, Sec. 6.1]). The $(k,2)$-quasicontinuous representative is unique, in the sense that two $(k,2)$-quasicontinuous representatives of the same function $u \in H^k(\mathbb{R}^n)$ coincide $\mathrm{Cap}_{k,2}$-quasi-everywhere. In addition, if U is an open subset of $\mathbb{R}^n$, then $u \in H^k_0(U)$ if and only if for all multi-index $\alpha \in \mathbb{N}^n$ with length $|\alpha| \leqslant k$, $\partial^\alpha u$ has a $(k-|\alpha|,2)$-quasicontinuous representative that vanishes $\mathrm{Cap}_{k-|\alpha|,2}$-quasi everywhere on ∂U, i.e., outside a set of zero $\mathrm{Cap}_{k-|\alpha|,2}$-capacity (see [1, Th. 9.1.3]). In the sequel, we will only be interested to the cases $k = 1$ or $k = 2$ in dimension $n = 2$.

2.3. Kondrat'ev spaces. — Following [25, Sec. 6.1], if C is an open cone of $\mathbb{R}^n$ with vertex at the origin, we define for any $\beta \in \mathbb{R}$ and $\ell \geqslant 0$ the weighted Sobolev space $V^\ell_\beta(C)$ by the closure of $\mathscr{C}^\infty_c(\overline{C} \smallsetminus \{0\})$ with respect to the norm

$$\|u\|_{V^\ell_\beta(C)} := \Big(\int_C \sum_{|\alpha| \leqslant \ell} |x|^{2(\beta-\ell+|\alpha|)} |\partial^\alpha u(x)|^2 dx\Big)^{1/2}.$$

It will also be useful to introduce the spaces $V^\ell_\beta(C)$ for $\ell < 0$, which is defined as the dual space of $V^{-\ell}_{-\beta}(C)$, endowed with the usual dual norm.

Observe that when $\ell \geqslant 0$ then $u \in V^\ell_\beta(C)$ if and only if the function $x \mapsto |x|^{\beta-\ell+|\alpha|}\partial^\alpha u(x) \in L^2(C)$ for all $|\alpha| \leqslant \ell$. If one is interested in homogeneous functions, it turns out that the parameter β plays a different role regarding to the integrability at the origin or at infinity. To fix the ideas, one can check that in dimension 2, a function of the form $x \mapsto |x|^\gamma f(x/|x|)$ around the origin and with compact support belongs to $V^\ell_\beta(C)$ for every $\beta < 1 - \gamma$. On the other hand, a function having this behavior at infinity and vanishing around the origin will belong to a space $V^\ell_\beta(C)$ for every $\beta > 1 - \gamma$. For instance if $\gamma = 3/2$, then the corresponding space of critical exponent would be that with $\beta = -1/2$.

2.4. Functions with Lebesgue deformation. — Given a vector field (distribution) $u : U \to \mathbb{R}^n$, the symmetrized gradient of u is denoted by

$$e(u) := \frac{\nabla u + \nabla u^T}{2}.$$

In linearized elasticity, u stands for the displacement, while $e(u)$ is the elastic strain. The elastic energy of a body is given by a quadratic form of $e(u)$ so that it is natural to consider displacements such that $e(u) \in L^2(U; \mathbb{M}^{n\times n}_{\rm sym})$. If U has Lipschitz boundary, it is well known that u actually belongs to $H^1(U;\mathbb{R}^n)$ as a consequence of Korn's inequality (see e.g. [10, 31]). However, when U is not smooth, we can only assert that $u \in L^2_{\rm loc}(U;\mathbb{R}^n)$. This motivates the following definition of the space of Lebesgue deformations:

$$LD(U) := \{u \in L^2_{\rm loc}(U;\mathbb{R}^n) : e(u) \in L^2(U; \mathbb{M}^{n\times n}_{\rm sym})\}.$$

If U is connected and u is a distribution with $e(u) = 0$, then necessarily it is a rigid movement, i.e., $u(x) = Ax + b$ for all $x \in U$, for some skew-symmetric matrix $A \in \mathbb{M}^{n\times n}$ and some vector $b \in \mathbb{R}^n$. If, in addition, U has Lipschitz boundary, the following Poincaré-Korn inequality holds: there exists a constant $c_U > 0$ and a rigid movement r_U such that

$$\|u - r_U\|_{L^2(U)} \leqslant c_U \|e(u)\|_{L^2(U)}, \quad \text{for all } u \in LD(U). \tag{2.1}$$

According to [2, Th. 5.2, Exam. 5.3], it is possible to make r_U more explicit in the following way: consider a measurable subset E of U with $|E| > 0$, then one can take

$$r_U(x) := \frac{1}{|E|}\int_E u(y)\,dy + \Big(\frac{1}{|E|}\int_E \frac{\nabla u(y) - \nabla u(y)^T}{2}\,dy\Big)\Big(x - \frac{1}{|E|}\int_E y\,dy\Big),$$

provided the constant c_U in (2.1) also depends on E.